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2680 [March, 1918]. **Proposed by C. C. YEN, Tangshan, North China.**

The diagonals of a maximum parallelogram inscribed in an ellipse are conjugate diameters of the ellipse. (From Joseph Edward's *Elementary Treatise on Differential Calculus.*)

SOLUTION BY L. E. MENSENKAMP, Freeport, Illinois.

Let the equation of the ellipse be $x^2/a^2 + y^2/b^2 = 1$. Now, it is easily shown that a quadrilateral inscribed in an ellipse is a parallelogram when, and only when, the opposite vertices are symmetrical with respect to the origin. In other words, the diagonals of an inscribed parallelogram pass through the center of the ellipse. Let us denote one vertex of the parallelogram (which for convenience we may assume to be in the first quadrant) by $P_1 = (x_1, y_1)$. Let an adjacent vertex be $P_2 = (x_2, y_2)$. Then, the vertex opposite P_1 will be $P_3 = (-x_1, -y_1)$, and that opposite P_2 will be $P_4 = (-x_2, -y_2)$.

We may call P_2P_3 the base of the parallelogram. Its equation may be written

$$(y_1 + y_2)x - (x_1 + x_2)y - (x_1 + x_2)y_1 + (y_1 + y_2)x_1 = 0.$$

The equation in this form enables us to apply the usual formula for the distance from a point to a line to obtain the distance from P_1 to the line P_2P_3, which is the altitude of the parallelogram. This expression for the altitude, after some reduction, becomes

$$D = \frac{2(x_1y_2 - x_2y_1)}{\sqrt{(y_1 + y_2)^2 + (x_1 + x_2)^2}}. \tag{1}$$

Multiplying D by the length of the base P_2P_3, we find the area of the parallelogram to be $A = 2(x_1y_2 - x_2y_1)$. Making use of the fact that these points lie on the ellipse, and assuming for the moment that P_2 lies above the X-axis, the area becomes

$$A = \frac{2b}{b}(x_1\sqrt{a^2 - x_2^2} - x_2\sqrt{a^2 - x_1^2}).$$

The condition for a maximum is that $\partial A/\partial x_1 = 0$, and $\partial A/\partial x_2 = 0$. Both of these conditions lead to the same equation, namely,

$$x_1x_2 = -\sqrt{a^2 - x_1^2}\sqrt{a^2 - x_2^2}. \tag{2}$$

Therefore,

$$\frac{y_1y_2}{x_1x_2} = \frac{y_1y_2}{-\sqrt{a^2 - x_1^2}\sqrt{a^2 - x_2^2}} = -\frac{b^2}{a^2}.$$

The last member of this equation follows from the elimination of y_1 and y_2 by means of the equation for the ellipse. Since y_1/x_1 and y_2/x_2 represent the slopes of the diagonals of the parallelogram, and since their product equals $-b^2/a^2$, it is seen (Bôcher, *Plane Analytic Geometry*, p. 153) that the diagonals are conjugate diameters of the ellipse.

If P_2 is assumed to be below the X-axis, we must remember to use

$$y_2 = -\frac{b}{a}\sqrt{a^2 - x_2^2},$$

but the final result is the same. It is evident that the equation (2) gives a maximum, and not a minimum, parallelogram; for, assuming P_2 temporarily fixed while P_1 varies, we see that the area is zero when P_1 coincides with either P_2 or P_4.

Also solved by J. B. REYNOLDS.

2681 [March, 1918]. **Proposed by PHILIP FRANKLIN, College of the City of New York.**

Given n letters of one kind and $n - 1$ letters of another kind, in how many ways can they be arranged so that, moving along the arrangement from one end to the other, the number of letters of the first kind passed over is greater than the number of the second kind at any instant?

Solution by C. F. Gummer, Queen's University.

Let $f(p, q)$ be the number of ways of arranging p letters, some A and some B, so that, on going from left to right, we pass *always* more A's than B's, and *finally* q more A's than B's. Clearly $f(p, q) = 0$ when $p - q$ is odd and when $q < 1$.

Such an arrangement will end with an A in $f(p-1, q-1)$ cases, $(p > 1)$; and it will end with a B in $f(p-1, q+1)$ cases $(q \neq 0)$. Therefore,

$$f(p, q) = f(p-1, q-1) + f(p-1, q+1)$$

under the above restrictions.

To remove the restriction $q \neq 0$, let us define

$$g(p, q) = \quad f(p, \quad q), \ (q \geqq 0);$$

$$g(p, q) = -f(p, -q), \ (q < 0).$$

Then

$$g(p, q) = g(p-1, q-1) + g(p-1, q+1)$$

provided $p > 1$.

By successive application,

$$g(p, q) = g(p-2, q-2) + 2q(p-2, q) + g(p-2, q+2) = g(p-3, q-3)$$
$$+ 3g(p-3, q-1) + 3g(p-3, q+1) + g(p-3, q+3) = \cdots = g(1, q-p+1)$$
$$+ \binom{p-1}{1} g(1, q-p+3) + \binom{p-1}{2} g(1, q-p+5) + \cdots.$$

Now $g(1, q) = 0$, except that $g(1, 1) = 1$ and $g(1, -1) = -1$.

Hence, if $p - q$ is even,

$$g(p, q) = -\binom{p-1}{(p-q-2)/2} + \binom{p-1}{(p-q)/2}$$
$$= \frac{2q}{p+q}\binom{p-1}{(p-q)/2}.$$

For the particular problem, $p = 2n - 1$, $q = 1$, and

$$f(p, q) = g(p, q) = \frac{1}{n}\binom{2n-2}{n-1} = \frac{\underline{|2n-2}}{\underline{|n}\,\underline{|n-1}}.$$

2683 [March, 1918]. Proposed by J. R. HITT, Mississippi College.

The height of a frustum of a cone is h, the radii of the upper and lower circular bases are a and b, respectively. Deduce the formula for finding the center of gravity of the frustum.

Solution by H. C. Gossard, U. S. Naval Academy.

Since the center of gravity is, obviously, on the axis of the frustum, let y be the required distance of the center of gravity above the lower base. Complete the cone and let x be the altitude of the upper cone. From similar triangles, $x : x + h = a : b$; whence, by division, $x : h = a : b - a$, or $x = ah/(b - a)$. The altitude of the entire cone is $x + h = bh/(b - a)$.

The volume of the whole cone is $\frac{\pi}{3}\frac{hb^3}{(b-a)} = V_1$ and the volume of the upper cone is $\frac{\pi}{3}\frac{ha^3}{(b-a)} = V_2$.

Hence, the volume of the frustum is $\frac{\pi}{3}\frac{hb^3}{(b-a)} - \frac{\pi}{3}\frac{ha^3}{(b-a)} = \frac{\pi}{3}h(a^2 + b^2 + ab) = V_3$.

Remembering that the center of gravity of a cone is on its axis of symmetry and 1/4 of the distance from the center of gravity of the base to the vertex measured from the base, we have, on taking moments about any diameter of the base, $V_3 y = \frac{1}{4}bh/b - aV_1 - \frac{1}{4}ah/b - aV_2$. Substituting the values of V_1, V_2, and V_3 and solving for $\bar{y}$ and reducing to simplest form, we have

$$y = \frac{h}{4}\left(\frac{b^2 + 2ab + 3a^2}{b^2 + ab + a^2}\right).$$